

DUALITY STRUCTURES AND DISCRETE CONFORMAL VARIATIONS OF PIECEWISE CONSTANT CURVATURE SURFACES

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ABSTRACT. A piecewise constant curvature manifold is a triangulated manifold that is assigned a geometry by specifying lengths of edges and stipulating that for a chosen background geometry (Euclidean, hyperbolic, or spherical), each simplex has an isometric embedding into the background geometry with the chosen edge lengths. Additional structure is defined either by giving a geometric structure to the Poincaré dual of the triangulation or by assigning a discrete metric, a way of assigning length to oriented edges. This notion leads to a notion of discrete conformal structure, generalizing the discrete conformal structures based on circle packings and their generalizations studied by Thurston and others. We define and analyze conformal variations of piecewise constant curvature 2-manifolds, giving particular attention to the variation of angles. We give formulas for the derivatives of angles in each background geometry, which yield formulas for the derivatives of curvatures. Our formulas allow us to identify particular curvature functionals associated with conformal variations. Finally, we provide a complete classification of discrete conformal structures in each of the background geometries.

1. INTRODUCTION

A triangulation of a manifold can be given a geometric structure by assigning compatible geometric structures to its component simplices. One of the easiest ways of doing this is to assign constant curvature geometries to the simplices, as these simplices are uniquely determined by their edge lengths. Such a structure gives a finitely parametrized set of geometric structures on a closed manifold.

In Thurston's formulation of the discrete Riemann mapping problem (see [43]) as well as in applied methods such as discrete exterior calculus (see, e.g., [16], [15]), it is important to not only have a piecewise constant curvature metric assigned to simplices, but also to give a structure to the Poincaré dual of the triangulation. Such structures arise naturally as incircle duals in Thurston's formulation of circle packings and as circumcentric duals in discrete exterior calculus. For piecewise Euclidean surfaces and 3-manifolds, in [22] and [24] the first author gives an axiomatic treatment of geometric duality structures that have orthogonal intersections with the primal simplices, and also relates these to discrete conformal variations.

The goal of the present work is to make precise the parametrization of duality structures by partial edge lengths (giving a discrete analogue of a Riemannian metric), define the general form of discrete conformal structures based on an axiomatic

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development related to conformal variation of angle, and derive a local classification of such structures. The relationship between duality structures and discrete metrics requires some understanding of possible geometric centers for triangles, leading to the definition of the span of a triangle as the space of possible geometric centers. The axiomatic development of conformal structure follows that in [24] for piecewise Euclidean surfaces, while the construction in piecewise hyperbolic and spherical surfaces is new. The general formulas for angle and curvature variation of piecewise hyperbolic and spherical surfaces is new (however, see the parallel work in [48]), generalizing circle packings and other discrete conformal structures previously studied by many authors (see Section 1.3 for details). The local classification of discrete conformal structures, giving explicit formulas for the structures, is new for each geometry including Euclidean.

We will begin by making these geometric structures precise, and then give precise statements of the main results.

1.1. Geometric structures on triangulations. In this section, we make precise some geometric structures.

Definition 1. *A triangulated manifold (M, T) is a topological manifold M together with a triangulation T of M . A (triangulated) piecewise constant curvature manifold (M, T, ℓ) with background geometry \mathbb{G} is a triangulated manifold (M, T) together with a function ℓ on the edges of the triangulation such that each simplex can be embedded in \mathbb{G} , a space of constant curvature, as a (nondegenerate) simplex with edge lengths determined by ℓ .*

When the background geometry is Euclidean ($\mathbb{G} = \mathbb{E}$), hyperbolic ($\mathbb{G} = \mathbb{H}$), or spherical ($\mathbb{G} = \mathbb{S}$), we call such a manifold piecewise flat, piecewise hyperbolic, or piecewise spherical, respectively.

When the background geometry is clear from context, we may omit it. Note that part of the definition is that the simplices are nondegenerate; this places inequality restrictions on the possible edge lengths. For instance, in Euclidean background the restrictions can be derived from Cayley-Menger determinants.

We will use $V = V(T)$ to denote the vertices in triangulation T and label them with numbers or letter such as $i \in V$. We will use $E = E(T)$ to denote edges and label them as a set of vertices $\{i, j\} \in E$, although most of this work could allow multiple edges between the same vertices or edges between the same vertex. We will use $E_+ = E_+(T)$ to denote oriented edges and label them with ordered pairs $(i, j) \in E_+$. Triangles will be denoted as a set of vertices, such as $\{i, j, k\}$. In a piecewise constant curvature manifold, the angle at vertex i in a triangle $\{i, j, k\}$ will be denoted γ_i . The set of real valued functions on V or E_+ will be denoted by V^* and E_+^* , respectively. We will use $\sigma < \tau$ to mean that σ is a subsimplex of τ .

1.1.1. Duality structures. The idea of a duality structure is that, in addition to the metric structure of a piecewise constant curvature manifold, we can put a geometric structure on the Poincaré dual cell complex by introducing geometric centers for pieces of the dual complex. Motivated by the Euclidean background case, we see that these geometric centers do not have to be constrained to the simplex, but its affine span. In the more general constant curvature case, we will need an analogue of the affine span that defines the space of possible simplex centers.

Since a piecewise constant curvature manifold is subdivided by simplices that can be embedded into the space \mathbb{G} , each simplex σ^k has a span defined as follows. First we need to define the underlying space of the span in each geometry.

Definition 2. *Given a constant curvature geometry \mathbb{G} , we define $\hat{\mathbb{G}}$ as follows:*

- *If $\mathbb{G} = \mathbb{E}^n$ then we take $\hat{\mathbb{G}}$ to be the underlying space \mathbb{R}^n .*
- *If $\mathbb{G} = \mathbb{H}^n$ then we take $\hat{\mathbb{G}}$ to be the entire space of the Klein model, also described as the extended hyperbolic plane in [9]. Note that in this case, $\mathbb{H}^n \subset \hat{\mathbb{H}}^n$.*
- *If $\mathbb{G} = \mathbb{S}^n$ then we take $\hat{\mathbb{G}}$ to be the quotient \mathbb{RP}^n of the sphere.*

We note the following easy facts about $\hat{\mathbb{G}}$.

- In each case, the isometry group of \mathbb{G} acts on $\hat{\mathbb{G}}$.
- In each case, there is a notion of orthogonality between two vectors, induced from the Euclidean dot product in the cases of Euclidean space and the sphere, and the Lorentzian bilinear product using the hyperboloid model of hyperbolic space and projecting to the Klein model space.
- In each case, any two points in $\hat{\mathbb{G}}$ can be connected by a line.

In what follows, we will assume that any simplex modeled on geometry \mathbb{G} can be isometrically embedded into $\hat{\mathbb{G}}$, and that embedding is unique up to isometry of \mathbb{G} . Note that, in the case of spherical geometry, the fact that a simplex embeds is a restriction on how big it can be. We are now ready to define the span.

Definition 3. *Given a simplex σ^k and an isometric embedding $\phi : \sigma^k \rightarrow \hat{\mathbb{G}}$, the span of σ^k under ϕ , denoted $S_\phi \sigma^k$, is the set*

$$S_\phi \sigma^k = \bigcup_{\substack{p, q \in \phi(\sigma^k) \\ p \neq q}} L_{p, q}$$

where $L_{p, q} \subset \hat{\mathbb{G}}$ is the line through the points p and q .

The span of σ^k , denoted $S\sigma^k$, is the quotient space obtained from the disjoint union $\bigsqcup_\phi S_\phi \sigma^k$ by identifying each pair of summands $S_\phi \sigma^k$ and $S_\rho \sigma^k$ by an isometry of \mathbb{G} that agrees with $\rho \circ \phi^{-1}$.

We remark that our definition is analogous to the definition of affine span in polytope theory (c.f., [34]). In both definitions, the span is viewed as a (geodesic) hyperplane tangent to the simplex/polytope-face as it sits in the ambient geometry.

The span has the property that for any points $x \in \sigma \subset S\sigma$ and $y \in S\sigma$, there is a unique line between x and y in $S\sigma \cong \hat{\mathbb{G}}$. The span also has the property that if $\sigma < \sigma'$ then there is a natural way in which $S\sigma \subset S\sigma'$.

Definition 4. *Suppose (M, T, ℓ) is a piecewise constant curvature manifold with background geometry \mathbb{G} .*

A duality structure for (M, T) is a choice of one point $C[\sigma] \in S\sigma$ from each simplex σ^k of T , subject to:

- *If $\sigma^\ell < \sigma^k$ then for any simplex $\tau = \{C[\sigma^\ell], C[\sigma^{\ell+1}], \dots, C[\sigma^k]\}$, we have that $S\tau$ is orthogonal to $S\sigma^\ell$ intersecting only at $C[\sigma^\ell]$.*

We say a duality structure is proper if it has Euclidean or spherical background or has hyperbolic background and the center of each edge is in \mathbb{H} .

Notice that in the case of spherical background, the centers lie in \mathbb{RP}^n and so correspond to two points in \mathbb{S}^n . We will often consider the span as \mathbb{S}^n with pairs of points instead of \mathbb{RP}^n . Proper duality structures are ones such that edge centers are determined by signed distances from the vertices, as determined by the partial edge lengths in the next section.

In general, we will denote the center of edge $\{i, j\}$ by c_{ij} and the center of triangle $\{i, j, k\}$ by c_{ijk} . These centers determine edge heights.

Definition 5. *Given a proper duality structure on a triangle $\{i, j, k\}$, each edge $\{i, j\}$ has a corresponding edge height h_{ij} determined by one of the following:*

- *If the center c_{ijk} is in the same half plane determined by the span $S\{i, j\}$ as the simplex $\{i, j, k\}$ is, h_{ij} is the distance between c_{ij} and c_{ijk} .*
- *If the center c_{ijk} is not in the same half plane determined by the span $S\{i, j\}$ as the simplex $\{i, j, k\}$ is, h_{ij} is the negative of the distance between c_{ij} and c_{ijk} .*
- *If the center c_{ijk} is in $\hat{\mathbb{H}}$ but not in \mathbb{H} , then the height is the distance from c_{ij} to c_{ijk}^\perp (see Section 3) with the same sign convention.*

1.1.2. Discrete metric structure. The definition of duality structure requires choosing centers. For a more explicit parametrization, we will try to adjust the metric structure ℓ in some way to ensure a duality structure. This is the role of metrics and pre-metrics.

The notion of a pre-metric is to reassign parts of the length function to the vertices. This is motivated partly by the definition of Riemannian metrics as tensor valued functions of the points of a manifold.

Definition 6. *Let (M, T) be a triangulated manifold. A pre-metric is an element $d \in E_+(T)^*$ such that (M, T, ℓ) is a piecewise constant curvature manifold with background geometry \mathbb{G} for the assignment $\ell_{ij} = d_{ij} + d_{ji}$ for every edge $\{i, j\}$.*

The d_{ij} are sometimes called partial edge lengths, since one considers the edge $\{i, j\}$ divided into two partial edges of length d_{ij} and d_{ji} . If the partial edge lengths are nonnegative, there is a point on the edge that is distance d_{ij} from vertex i and distance d_{ji} from vertex j , and this point is called the edge center. Note that if one of the partial edge lengths is negative, there is an interpretation in terms of signed distance, and there is still a center, this time on the span of the edge.

We would like to restrict pre-metrics to those that generate geometries on the Poincaré dual structure such that dual and primal cells intersect orthogonally. If one considers the point c_{ij} on the span of an edge $\{i, j\}$ that is distance d_{ij} from vertex i and d_{ji} from vertex j (distance can be considered with sign so one partial edge length can be negative), a center is determined. One can consider the plane orthogonal to the span $S\{i, j\}$ through c_{ij} , and use the intersections of these planes to construct more centers (e.g., if the planes of the three edges of a triangle intersect at a point then we use that point as the center of the triangle). This construction is explained in detail for Euclidean background in [22]. We wish to characterize which conditions on the pre-metrics guarantee that these centers exist and give a duality structure. We call these metrics, and the actual motivations for the following definitions are characterization theorems given later. The main advantage of metrics over duality structures is that the metrics entirely parametrize the geometry, and so the space of metrics is relatively easy to describe.

Definition 7. A discrete metric, or metric, on (M, T) with background geometry \mathbb{G} is a pre-metric d such that for every triangle $\{i, j, k\}$ in T ,

$$(1.1) \quad d_{ij}^2 + d_{jk}^2 + d_{ki}^2 = d_{ji}^2 + d_{kj}^2 + d_{ik}^2 \quad \text{if } \mathbb{G} = \mathbb{E},$$

$$(1.2) \quad \cosh(d_{ij}) \cosh(d_{jk}) \cosh(d_{ki}) = \cosh(d_{ji}) \cosh(d_{kj}) \cosh(d_{ik}) \quad \text{if } \mathbb{G} = \mathbb{H},$$

$$(1.3) \quad \cos(d_{ij}) \cos(d_{jk}) \cos(d_{ki}) = \cos(d_{ji}) \cos(d_{kj}) \cos(d_{ik}) \quad \text{if } \mathbb{G} = \mathbb{S}.$$

A piecewise constant curvature, metrized manifold (M, T, d) with background geometry \mathbb{G} is a triangulated manifold (M, T) together with a metric d . We denote the space of all metrics with background geometry \mathbb{G} on a given triangulated manifold (M, T) by $\mathbf{met}_{\mathbb{G}}(M, T)$.

Note that the space of metrics $\mathbf{met}_{\mathbb{G}}(M, T)$ on a finite triangulation is determined as a subset of $\mathbb{R}^{|E+|}$ by a number of equalities of the form above (one for each triangle) and a number of inequalities (to ensure the simplices are nondegenerate).

1.1.3. Discrete conformal structure. A discrete conformal structure is a particular way of determining the metric from information assigned to points (vertices). It is partly motivated by this characterization of conformal change of a Riemannian metric, and also by Thurston's formulation of conformal circle packing structure. A general formulation for Euclidean background is described in [24], and there are a number of formulations of specific cases of analogous structures in hyperbolic and spherical backgrounds (see Section 1.3).

Based on Propositions 1, 6, and 10, if we suppose that the pre-metric is determined by weights on the vertex endpoints, there is a restriction that ensures that the resulting pre-metric is actually a discrete metric, i.e., it determines a duality structure. In addition, we want conformal structures to have nice formulas for angle variations. This motivates the following definition.

Definition 8. A discrete conformal structure $\mathcal{C}(M, T, U)$ on a triangulated manifold (M, T) with background geometry \mathbb{G} on an open set $U \subset V(T)^*$ is a smooth map

$$\mathcal{C}(M, T, U) : U \rightarrow \mathbf{met}_{\mathbb{G}}(M, T)$$

such that if $d = \mathcal{C}(M, T, U)[f]$ then for each $(i, j) \in E_+(T)$ and $k \in V(T)$,

$$(1.4) \quad \frac{\partial \ell_{ij}}{\partial f_i} = d_{ij} \quad \text{if } \mathbb{G} = \mathbb{E},$$

$$(1.5) \quad \frac{\partial \ell_{ij}}{\partial f_i} = \tanh d_{ij} \quad \text{if } \mathbb{G} = \mathbb{H},$$

$$(1.6) \quad \frac{\partial \ell_{ij}}{\partial f_i} = \tan d_{ij} \quad \text{if } \mathbb{G} = \mathbb{S},$$

and

$$\frac{\partial d_{ij}}{\partial f_k} = 0$$

if $k \neq i$ and $k \neq j$.

A conformal variation of a metric $d = \mathcal{C}(M, T, U)[f]$ is the change of the metric in the conformal class as f changes, and is determined by derivatives such as $\partial d_{ij} / \partial f_i$.

We have chosen the parameter f so that the variation formulas above are as simple as possible. However, we will sometimes choose to parametrize the structures differently (see Theorem 3). Also note that with conformal variations, the choice of the set U is not particularly important; we only need the existence of a neighborhood around any point in U .

1.2. Main theorems. In this paper, we study the relationships between duality structures, discrete metrics, and conformal variations. The main new contributions are the following: (1) a characterization of duality structures in hyperbolic and spherical backgrounds, generalizing the notion of length structures arising from circles with given radii and inversive distances, (2) calculation of the conformal variation of angles in a triangle for hyperbolic and spherical backgrounds together with determining a functional making the curvature variational, and (3) a classification theorem for discrete conformal variations of Euclidean, hyperbolic, and spherical triangles, including the formulation of the notion of discrete conformal variations from basic principles.

1.2.1. Equivalence of duality and metric structures. The following theorem characterizes duality structures on surfaces in each of the constant curvature backgrounds.

Theorem 1. *Let (M, T, ℓ) be a piecewise constant curvature 2-manifold. There is a one-to-one correspondence between proper duality structures on (M, T, ℓ) and discrete metric structures on (M, T, ℓ) .*

This theorem follows from Propositions 1, 6, and 10.

1.2.2. Discrete conformal variations of angle. The following theorem gives the variation of angle formulas. The Euclidean result is in [24], and the hyperbolic and spherical results are new (compare [48]).

Theorem 2. *For any conformal variation of a metric $d = \mathcal{C}(M, T, U)[f]$ with background geometry \mathbb{G} of a surface M^2 , we have for any edge $\{i, j\}$ the following formulas.*

- In Euclidean background,

$$(1.7) \quad \frac{\partial \gamma_i}{\partial f_j} = \frac{h_{ij}}{\ell_{ij}}$$

$$(1.8) \quad \frac{\partial \gamma_i}{\partial f_i} = -\frac{h_{ij}}{\ell_{ij}} - \frac{h_{ik}}{\ell_{ik}}.$$

- In hyperbolic background,

$$(1.9) \quad \frac{\partial \gamma_i}{\partial f_j} = \frac{1}{\cosh d_{ji}} \frac{\tanh^\beta h_{ij}}{\sinh \ell_{ij}}$$

$$(1.10) \quad \frac{\partial \gamma_i}{\partial f_i} = -\frac{1}{\cosh d_{ji}} \frac{\tanh^\beta h_{ij}}{\tanh \ell_{ij}} - \frac{1}{\cosh d_{ki}} \frac{\tanh^\beta h_{ik}}{\tanh \ell_{ik}}$$

where β is 1 if c_{ijk} is timelike and -1 if c_{ijk} is spacelike.

- In spherical background,

$$(1.11) \quad \frac{\partial \gamma_i}{\partial f_j} = \frac{1}{\cos d_{ji}} \frac{\tan h_{ij}}{\sin \ell_{ij}}$$

$$(1.12) \quad \frac{\partial \gamma_i}{\partial f_i} = -\frac{1}{\cos d_{ji}} \frac{\tan h_{ij}}{\tan \ell_{ij}} - \frac{1}{\cos d_{ki}} \frac{\tan h_{ik}}{\tan \ell_{ik}}$$

This theorem follows from Theorems 5, 7, and 9 together with Propositions 9 and 11.

It turns out that although the variables f for the conformal variations are quite natural, a change of variables gives that the curvatures are the gradient of a functional, where the curvatures are defined as

$$K_i = 2\pi - \sum_{\{i,j,k\}} \gamma_i$$

for each vertex i , where the sum is over all triangles containing i .

Theorem 3. *Consider a piecewise constant curvature, metrized 2-manifold (M, T, d) , where $d = d(f)$ is determined by a conformal structure. There is a change of variables $u = u(f)$ such that*

$$\frac{\partial \gamma_i}{\partial u_j} = \frac{\partial \gamma_j}{\partial u_i}$$

and hence if we fix a \bar{u} there is a functional

$$F = 2\pi \sum_{i \in V} u_i - \sum_{\{i,j,k\}} \int_{\bar{u}}^u (\gamma_i du_i + \gamma_j du_j + \gamma_k du_k)$$

with the property that

$$\frac{\partial F}{\partial u_i} = K_i.$$

Furthermore, if all $d_{ij} > 0$ and $h_{ij} > 0$ and then this function is strictly convex if $\mathbb{G} = \mathbb{H}$ and weakly convex (strictly convex except for scaling) if $\mathbb{G} = \mathbb{E}$.

This theorem follows from Theorems 6, 8, and 10.

1.2.3. Classification of discrete conformal structures. The following theorems classify discrete conformal variations in each of the constant curvature backgrounds. The results are new for all background geometries.

Theorem 4. *Let $\mathcal{C}(M, T, U)$ be a discrete conformal class with background geometry \mathbb{G} on a surface M . Then there exist $\alpha \in \mathbb{R}^{|V|}$ and $\eta \in \mathbb{R}^{|E|}$ such that the conformal structure can be written as*

$$d_{ij} = \frac{\alpha_i e^{2f_i} + \eta_{ij} e^{f_i + f_j}}{\ell_{ij}}$$

with

$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i + f_j}.$$

if $\mathbb{G} = \mathbb{E}$,

$$\tanh d_{ij} = \frac{\alpha_i e^{2f_i}}{\sinh \ell_{ij}} \sqrt{\frac{1 + \alpha_j e^{2f_j}}{1 + \alpha_i e^{2f_i}}} + \frac{\eta_{ij} e^{f_i + f_j}}{\sinh \ell_{ij}}$$

with

$$\cosh \ell_{ij} = \sqrt{(1 + \alpha_i e^{2f_i})(1 + \alpha_j e^{2f_j})} + \eta_{ij} e^{f_i + f_j}.$$

if $\mathbb{G} = \mathbb{H}$, or

$$\tan d_{ij} = \frac{\alpha_i e^{2f_i}}{\sin \ell_{ij}} \sqrt{\frac{1 - \alpha_j e^{2f_j}}{1 - \alpha_i e^{2f_i}}} + \frac{\eta_{ij} e^{f_i + f_j}}{\sin \ell_{ij}}$$

with

$$\cos \ell_{ij} = \sqrt{(1 - \alpha_i e^{2f_i})(1 - \alpha_j e^{2f_j})} - \eta_{ij} e^{f_i + f_j}.$$

if $G = S$.

This theorem is proven in each case in Sections 2.3, 5.2, and 6.

In light of Theorem 4, one can also calculate angle variations from Theorem 2 based on the conformal structures determined by α and η . These conformal structures are sometimes referred to as $\mathcal{C}_{\alpha, \eta}$ (see, e.g., [25]).

1.3. Comparison with previous formulations. In this section we briefly compare our parametrizations with other parametrizations of certain discrete conformal structures. The formulation in this paper unifies the previous work into a single formula for each background geometry and generalizes some of these. Independently, [48] derived a formula for the variation of angle that is essentially the same as ours, though we express it and prove it in a different way. We note that the Euclidean background case was treated in [24], which also describes the relationship of the general case to previous formulations.

The first formulation of the circle packing conformal structure (corresponding, in our notation, to $\alpha_i = 1$ and $\eta_{ij} = 1$ for all vertices and edges) is in Thurston's work [44]. Many of the relevant calculations are followed through in [35], and the first variational formulation is due to Colin de Verdière in [12]. In each of these cases, the Euclidean and hyperbolic cases were treated, and the conformal structures were either circles with given intersection angles between 0 and $\pi/2$ (corresponding, in our notation, to $\alpha_i = 1$ and $0 \leq \eta_{ij} \leq 1$ for all vertices and edges). Additional work was done by Chow-Luo in [10]. The case of circles with fixed inversive distances (corresponding, in our notation, to $\alpha_i = 1$ and $|\eta_{ij}| \geq 1$ for all vertices and edges) was introduced by Bowers and Stephenson [4] and the variational perspective was pursued by Guo in [26] (this was anticipated by Springborn's work on volumes of hyperideal simplices in [41]).

The multiplicative conformal structure (corresponding, in our notation, to $\alpha_i = 0$ for all vertices) was apparently first suggested in [40], but most of the mathematical ideas arose in work of Luo [33] and Springborn-Schrader-Pinkall [42] in the Euclidean case. Generalizing to the hyperbolic case was not obvious, but work in this direction first appeared in work by Bobenko-Pinkall-Springborn [1]. It is notable that the proper parametrization variable is not clear in this case, and this issue is discussed in Section 5.3. The unified case for Euclidean background is given in [24] and the hyperbolic case was first described in this paper and independently in [48]. For more on some of these discrete conformal structures, see the books [43], [14], and [47].

Explicit calculation of the variation of angle coefficients in the Euclidean circle packing case is due to Z. He [28], and followed by the first author in [24]. The coefficients are closely related to the discrete Laplacians found in [18], [11], [36], [17], [3], [23], [16], [29], [45], [46], and many other places.

There are close connections between these variational viewpoints and hyperbolic volumes, as evidenced by work of Brägger [5], Rivin [39], Garret [20], Leibon [32], Bobenko-Springborn [2], Springborn [41], Springborn-Schrader-Pinkall [42], and Bobenko-Pinkall-Springborn [1], Fillastre-Izmestiev [19], and Zhang et. al. [48].

Some of this work was generalized to discrete conformal structures in three dimensions by Cooper-Rivin in [13] and the first author in [21] and [24]. While the

functionals whose variations lead to curvatures in two dimensions are possibly related to the log determinant of the Laplacian and surface entropy (see [32]), in three dimensions the functional is related to Regge's formulation of the Einstein-Hilbert (total scalar curvature) functional. See, e.g., [38], [7], [27], [6], [30], [31].

2. EUCLIDEAN GEOMETRY

2.1. Duality structures on Euclidean triangles. Clearly, the choice of a pre-metric with Euclidean background determines the geometry of each triangle $\{i, j, k\}$ and for any isometric embedding, specifies the triangle's sides $\{e_{ij}\}$ with lengths $\{\ell_{ij}\}$. Through each finite edge e_{ij} of the triangle we have a unique line E_{ij} , considered in \mathbb{E} .

Suppose we identify E_{ij} with the real number line such that v_i is at the origin and v_j is on the positive x axis. Given these coordinates, we specify the edge centers $c_{ij} = c_{ji} = C(\{i, j\})$ to be the point d_{ij} on the line. Note that d_{ji} denotes the distance between c_{ij} and v_j , considered with a sign determined by which side of v_j in E_{ij} contains c_{ij} .

For each edge $\{i, j\}$, there exists a unique line P_{ij} that passes through c_{ij} and is orthogonal to E_{ij} .

In [22] (Proposition 4), the first author presented a necessary and sufficient condition on the partial edges to guarantee the three lines $\{P_{ij}\}$ meet at a single point:

Proposition 1. *Suppose $\{d_{ij}\}$ is a Euclidean pre-metric. Then the perpendiculars $\{P_{ij}\}$ meet at a single point if and only if*

$$(2.1) \quad d_{12}^2 + d_{23}^2 + d_{31}^2 = d_{21}^2 + d_{32}^2 + d_{13}^2.$$

This motivates the Euclidean case of Definition 7 and proves the Euclidean case of Theorem 1.

2.2. Conformal variation of angle. The conformal structure is defined in such a way as to give the following variational formula.

Theorem 5. *Given a conformal structure, we have*

$$\frac{\partial \gamma_i}{\partial f_j} = \frac{h_{ij}}{\ell_{ij}}$$

if $i \neq j$ and

$$\frac{\partial \gamma_i}{\partial f_i} = -\frac{h_{ij}}{\ell_{ij}} - \frac{h_{ik}}{\ell_{ik}}.$$

This theorem is proven in [22], generalizing the theorems in special cases given in [28] and [21]. It follows easily (see, e.g., [24]) that the curvature is variational with respect to a convex functional.

Theorem 6. *The partial derivatives of the angles in a triangle are symmetric, i.e.,*

$$\frac{\partial \gamma_i}{\partial f_j} = \frac{\partial \gamma_j}{\partial f_i}$$

and hence if we fix a \bar{f} there is a functional

$$F = 2\pi \sum_{i \in V} f_i - \sum_{\{i,j,k\}} \int_{\bar{f}}^f (\gamma_i df_i + \gamma_j df_j + \gamma_k df_k)$$

with the property that

$$\frac{\partial F}{\partial f_i} = K_i.$$

Furthermore, if all $d_{ij} > 0$ and $h_{ij} > 0$ and then this function is weakly convex (strictly convex except for scaling).

2.3. Characterization of discrete conformal structures. In this section we prove the characterization theorem. Recall that the only assumptions are:

- The compatibility condition 1.1 for the triangle with vertices v_i, v_j , and v_k .
- The assumption that d_{ij} depends only on f_i and f_j .

Proof of the Euclidean case of Theorem 4. We first note that

$$(2.2) \quad \begin{aligned} \frac{\partial \ell_{ij}^2}{\partial f_i} &= \ell_{ij}^2 + d_{ij}^2 - d_{ji}^2 \\ \frac{\partial \ell_{ij}^2}{\partial f_j} &= \ell_{ij}^2 - (d_{ij}^2 - d_{ji}^2) \end{aligned}$$

and that

$$\frac{\partial^2}{\partial f_i \partial f_j} (d_{ij}^2 - d_{ji}^2) = 0$$

since for any triangle with vertices v_i, v_j, v_k we have

$$d_{ij}^2 - d_{ji}^2 = d_{ik}^2 + d_{kj}^2 - d_{jk}^2 - d_{ki}^2.$$

We can compute that

$$\left(\frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j} \right) d_{ij} = \frac{\partial d_{ij}}{\partial f_i} + \frac{\partial d_{ij}}{\partial f_j} = \frac{\partial d_{ij}}{\partial f_i} + \frac{\partial d_{ji}}{\partial f_i} = d_{ij}$$

since

$$\frac{\partial d_{ij}}{\partial f_j} = \frac{\partial^2 \ell_{ij}}{\partial f_i \partial f_j} = \frac{\partial d_{ji}}{\partial f_i}.$$

It follows that

$$\left(\frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j} \right) (d_{ij}^2 - d_{ji}^2) = 2(d_{ij}^2 - d_{ji}^2)$$

and so it follows that

$$\frac{\partial^2}{\partial^2 f_i} (d_{ij}^2 - d_{ji}^2) = 2 \frac{\partial}{\partial f_i} (d_{ij}^2 - d_{ji}^2)$$

and

$$\frac{\partial^2}{\partial^2 f_j} (d_{ij}^2 - d_{ji}^2) = 2 \frac{\partial}{\partial f_j} (d_{ij}^2 - d_{ji}^2).$$

We can solve these equations, getting

$$\begin{aligned} \frac{\partial}{\partial f_i} (d_{ij}^2 - d_{ji}^2) &= 2a_{ij}e^{2f_i}, \\ \frac{\partial}{\partial f_j} (d_{ij}^2 - d_{ji}^2) &= -2a_{ji}e^{2f_j} \end{aligned}$$

for constants a_{ij} and a_{ji} . Hence

$$d_{ij}^2 - d_{ji}^2 = a_{ij}e^{2f_i} - a_{ji}e^{2f_j}.$$

We can now use (2.2) to find that for a constant η_{ij}

$$(2.3) \quad \ell_{ij}^2 = a_{ij}e^{2f_i} + a_{ji}e^{2f_j} + 2\eta_{ij}e^{f_i+f_j}.$$

From this, we compute that

$$d_{ij} = \frac{\partial \ell_{ij}}{\partial f_i} = \frac{a_{ij}e^{2f_i} + \eta_{ij}e^{f_i+f_j}}{\ell_{ij}}.$$

We note that in a triangle, since

$$d_{ij}^2 - d_{ji}^2 + d_{ki}^2 - d_{ik}^2 = d_{kj}^2 - d_{jk}^2$$

and the right side is independent of f_i , differentiating with respect to f_i gives

$$2(a_{ij} - a_{ik})e^{2f_i} = 0$$

and hence $a_{ij} = a_{ik}$ and a is independent of the edge, only depending on the vertex, hence we rename $\alpha_i = a_{ij} = a_{ik}$.

To see that the α_i and η_{ij} must be consistent across triangles, consider Equation 2.3 on both triangles and differentiate with respect to f_i and f_j to see that the η_{ij} agree and then f_i to see that the α_i agree. \square

3. BASIC CALCULATIONS IN HYPERBOLIC GEOMETRY

Before we move to the hyperbolic versions of the previous work, we will review some techniques for computing in hyperbolic geometry. This section summarizes the elementary facts about the hyperbolic plane \mathbb{H} that we will use in later calculations. All of the propositions in this section are discussed in Chapter 3 of [37]. See also [8]. For the reader's convenience, we have included some, but not all, proofs.

We use the hyperboloid model of \mathbb{H} for the majority of our calculations. In this model, the vector space \mathbb{R}^3 is equipped with a Lorentzian inner product $*$ given by $u * v := u^T J v$ where J is the diagonal matrix with entries 1,1,-1. We define a “hyperbolic magnitude” $\|u\| := \sqrt{u * u}$; the only possible hyperbolic lengths are nonnegative scalar multiples of 1 and i . \mathbb{H} corresponds to those vectors $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ satisfying $u * u = -1$ and $u_3 > 0$.

Definition 9. A vector $u \in \mathbb{R}^3$ is termed spacelike if $u * u > 0$, lightlike (or “on the light cone”) if $u * u = 0$, and timelike if $u * u < 0$.

The vector space structure on $(\mathbb{R}^3, *)$ gives us several ways to describe a geodesic in \mathbb{H} :

- As a nonempty intersection $\mathbb{H} \cap \text{Span}(p, q)$ for linearly independent $p, q \in \mathbb{R}^3$.
- As a nonempty intersection $\mathbb{H} \cap p^\perp$, where p is a spacelike vector and $p^\perp := \{v \in \mathbb{R}^3 : p * v = 0\}$.
- As a path, parametrized by arclength, given by $\gamma(t) = \cosh(t)p + \sinh(t)v$. In this form, $p \in \mathbb{H}$, $v \in p^\perp$ with $v * v = 1$. Note p and v encode the position and direction of γ at $t = 0$.

The second characterization becomes particularly useful when combined with the Lorentzian cross product, which is given by $p \otimes q := J(x \times y)$. Clearly, the Lorentzian cross product has two useful properties:

- $p \otimes q = 0$ if and only if p and q are linearly dependent.
- $p \otimes q$ is $*$ -orthogonal to both p and q .

A consequence of the second observation is that given distinct points $p, q \in \mathbb{H}$, one simple way to describe the geodesic through p and q is $(p \otimes q)^\perp$.

In the sequel, we will use $d_{\mathbb{H}}(u, v)$ to denote the hyperbolic distance between two timelike points, and $d_{\mathbb{H}}(u, v^\perp)$ to denote the hyperbolic distance between a timelike point and a geodesic in hyperbolic space determined as the orthogonal complement of a spacelike point. When $u, v \in \mathbb{R}^3$ satisfy $|u * v| = |v * u| = 1$, we have the following interpretations of the quantity $u * v$:

- If u and v are both timelike, then $u * v = -\cosh(d_{\mathbb{H}}(u, v))$.
- If u is timelike and v is spacelike, then $u * v = \pm \sinh(d_{\mathbb{H}}(u, v^\perp))$ and the sign depends upon which of the halfspaces bounded by v^\perp contains u .
- If u and v are both spacelike and u^\perp and v^\perp intersect in angle α within \mathbb{H} , $u * v = \cos(\alpha)$.

Notice that the last item implies that for spacelike u and v , u^\perp and v^\perp meet at a right angle if and only if $u * v = 0$.

The following identities simplify calculations that involve Lorentzian cross products. Suppose $x, y, z, w \in \mathbb{R}^3$:

$$(3.1) \quad x \otimes y = -y \otimes x,$$

$$(3.2) \quad (x \otimes y) * z = \det(x, y, z),$$

$$(3.3) \quad x \otimes (y \otimes z) = (x * y)z - (z * x)y,$$

$$(3.4) \quad (x \otimes y) * (z \otimes w) = \begin{vmatrix} x * w & x * z \\ y * w & y * z \end{vmatrix}.$$

We have already seen that several different kinds of data can be used to specify a geodesic on \mathbb{H} . This allows us to extend our understanding of where geodesics intersect.

Definition 10. *Given a geodesics γ on \mathbb{H} , we will identify γ with the unique 2-dimensional subspace P_γ of \mathbb{R}^3 such that $P_\gamma \cap \mathbb{H}$ is the image of γ .*

Given geodesics γ, ω on \mathbb{H} , we define their intersection to be their intersection as subspaces of \mathbb{R}^3 , namely $P_\gamma \cap P_\omega$.

Readers familiar with the Klein model of \mathbb{H} (the central projection of \mathbb{H} onto the plane $z = 1$) should note that this definition is simply a linear-algebraic way of formulating the notion of intersecting 1-hyperplanes in the Klein model.

Introducing a broader notion of intersection allows us to generalize familiar equations (like the law of cosines) and express them in terms of linear algebra. Understanding how to interpret the Lorentzian inner product is key to relating these different formulas. Often, the linear algebraic interpretation allows us to efficiently treat several seemingly different cases at once.

Recall the definition of a triangle (see Section 3.5 in [37]), which allows some of the vertices to be timelike, lightlike, or spacelike. We will concentrate on triangles with at least two timelike vertices.

Proposition 2. *Suppose $x \in \mathbb{H}$ and $y, z \in \mathbb{R}^3$ are either timelike or spacelike. Then*

$$(z \otimes x) * (x \otimes y) = -\|z \otimes x\| \cdot \|x \otimes y\| \cos(\alpha),$$

where α is the angle at x in the (clockwise oriented) triangle $\{x, y, z\}$.

Proposition 3 (The Generalized Law of Cosines). *Suppose $x, y, z \in \mathbb{R}^3$, with $\|x\| = \|z\| = i$ and $\|y\| = 1$ or i , are the vertices of a triangle in \mathbb{H} , with angle α at x . Then*

$$z * y + (z * x)(x * y) = \|z \otimes x\| \|x \otimes y\| \cos(\alpha).$$

Proof. Assume, without loss of generality, that x, y, z label the vertices of the triangle in clockwise order. Equation 3.4 implies

$$-(z \otimes x) * (x \otimes y) = (z * y) + (z * x)(x * y).$$

Now apply Proposition 2 to obtain the desired equality. \square

By setting $\alpha = \pi/2$, we obtain a generalized version of the Pythagorean theorem:

Corollary 1 (The Generalized Pythagorean Theorem). *Suppose x, y, z are the vertices of a right triangle, with the right angle at x . Then:*

$$-(z * y) = (z * x)(x * y).$$

We will require formulas for performing trigonometry in a hyperbolic right triangle where one of the vertices (not the one adjacent to the right angle) may be spacelike or timelike. Suppose we have a right triangle labeled like the one in Figure 1.

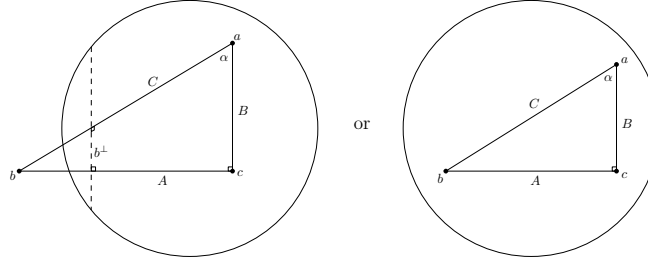


FIGURE 1. Two (Generalized) Right Triangles in the Klein Model

Proposition 4. *Given a triangle labeled as in Figure 1, we have:*

$$\cos(\alpha) = \frac{\tanh(B)}{\tanh(C)}, \quad \sin(\alpha) = \frac{\sinh(A)}{\sinh(C)}, \quad \tan(\alpha) = \frac{\tanh(A)}{\sinh(B)}$$

if b is timelike and

$$\cos(\alpha) = \tanh(B) \tanh(C), \quad \sin(\alpha) = \frac{\cosh(A)}{\cosh(C)}, \quad \tan(\alpha) = \frac{1}{\sinh(B) \tanh(A)}$$

if b is spacelike.

Deriving these formulas is an easy application of the generalized Pythagorean theorem and the generalized law of cosines.

The next corollary generalizes the familiar formula for the cosine of an angle in a hyperbolic right triangle.

Corollary 2. *Suppose x, y, z are the vertices of a right triangle (with the right angle at z) satisfying the assumptions of Proposition 3. Then*

$$\cos(\alpha) = -\frac{x * y}{\|x \otimes y\|} \tanh(d_{\mathbb{H}}(z, x)).$$

Proof. The right angle at z means that:

$$\begin{aligned} 0 &= (y \otimes z) * (z \otimes x) \\ &= -(y * x) - (y * z)(z * x) \end{aligned}$$

and so

$$z * y = -\frac{y * x}{z * x}.$$

Substituting this into the equation we obtain from the Law of Cosines, we learn:

$$\begin{aligned} \|z \otimes x\| \|x \otimes y\| \cos(\alpha) &= z * y + (z * x)(x * y) \\ &= -\frac{y * x}{z * x} + (z * x)(x * y) \\ &= (x * y) \frac{(z * x)^2 - 1}{z * x} \\ &= (x * y) \frac{\sinh^2(d_{\mathbb{H}}(z, x))}{-\cosh(d_{\mathbb{H}}(z, x))}. \end{aligned}$$

Using Equation 3.4, it is easy to check $\|z \otimes x\| = \sinh(d_{\mathbb{H}}(z, x))$. Hence:

$$\cos(\alpha) = -\frac{x * y}{\|x \otimes y\|} \tanh(d_{\mathbb{H}}(z, x)).$$

□

Because the Lorentzian inner product is nondegenerate, we have a well defined notion of $*$ -orthogonality and may apply the Gram-Schmidt procedure to obtain a basis of mutually $*$ -orthogonal vectors. This procedure can be used to parametrize a geodesic given in the form $\mathbb{H} \cap \text{Span}(p, q)$ by arclength.

Proposition 5. *Suppose $p \in \mathbb{H}$, and $q \in \mathbb{R}^3$. Then the geodesic $\mathbb{H} \cap \text{Span}(p, q)$ may be parametrized by arclength as:*

$$\gamma(t) = \cosh(t)p + \sinh(t) \frac{q + (p * q)p}{\sqrt{q * q + (p * q)^2}}.$$

Proof. The geodesic in question can be parametrized by arclength as $\gamma(t) = \cosh(t)p + \sinh(t)v$ for some spacelike v with $v * v = 1$; we simply need to use the Gram-Schmidt procedure to guarantee that $\text{Span}(p, v) = \text{Span}(p, q)$ and $v \in p^\perp$.

So consider the vector $q + (p * q)p$. Notice $-(p * q)p$ is the $*$ -projection of q onto the subspace spanned by p , and

$$p * (q + (p * q)p) = p * q - p * q = 0.$$

To find v , we only need to rescale this projection. Since

$$\begin{aligned} (q + (p * q)p) * (q + (p * q)p) &= q * q + 2(p * q)^2 + (p * q)^2(p * p) \\ &= q * q + (p * q)^2 \end{aligned}$$

the appropriate v is

$$v = \frac{q + (p * q)p}{\sqrt{q * q + (p * q)^2}}.$$

□

4. DUALITY STRUCTURES ON HYPERBOLIC TRIANGLES

We interpret a piecewise hyperbolic pre-metric as subdividing each edge $\{i, j\}$ of length ℓ_{ij} into two portions of length d_{ij} and d_{ji} , that are assigned to the vertices i and j respectively.

Definition 11. *Given a pre-metric d and an isometric embedding of a simplex $\{i, j\}$ into \mathbb{H} :*

- *The vertices $p_i, p_j \in \mathbb{H}$ of $\{i, j\}$ are the images of i and j under the embedding.*
- *The edge center c_{ij} induced by d is the unique point along the line E_{ij} through p_i and p_j such that c_{ij} is (signed) distance d_{ij} from p_i and d_{ji} from p_j .*
- *The edge perpendicular P_{ij} is the line through c_{ij} that is orthogonal to E_{ij} .*

Unlike in the Euclidean setting, it is possible that the geodesics P_{ij} and P_{jk} do not intersect within \mathbb{H} . However, these two 1-hyperplanes can be understood as intersecting in the more general sense of Definition 10, namely the two-dimensional subspaces of $(\mathbb{R}^3, *)$ associated to P_{ij} and P_{jk} intersect in a one-dimensional subspace. One can then ask for necessary and sufficient conditions on the pre-metric that guarantee that for each simplex $\{i, j, k\}$

$$(4.1) \quad P_{ij} \cap P_{jk} = P_{jk} \cap P_{ki} = P_{ki} \cap P_{ij}$$

or, colloquially, the three perpendiculars of $\{i, j, k\}$ intersect in a single point (this point is in the span of $\{i, j, k\}$). This condition can also be interpreted in the Klein model of hyperbolic space as the condition that the three lines representing the geodesics intersect at the same point in the plane of the Klein model.

Proposition 6. *Suppose d is a piecewise hyperbolic pre-metric. Equation 4.1 holds if and only if the following compatibility equation*

$$(4.2) \quad (p_i * c_{ij})(p_j * c_{jk})(p_k * c_{ki}) = (p_i * c_{ki})(p_j * c_{ij})(p_k * c_{jk})$$

is satisfied for every simplex $\{i, j, k\}$.

Since the vectors p_i and c_{ij} are timelike of length -1, Equation 4.2 has the following equivalent formulation:

$$\cosh(d_{ij}) \cosh(d_{jk}) \cosh(d_{ki}) = \cosh(d_{ji}) \cosh(d_{kj}) \cosh(d_{ik}).$$

Proof. To simplify our notation, we shall consider a single 2-simplex $\{1, 2, 3\}$. The vertices of the embedded 2-simplex are linearly independent vectors $p_1, p_2, p_3 \in \mathbb{H}$.

Consider that if c is a point on the perpendicular P_{ij} , then $P_{ij} = (c \otimes c_{ij})^\perp$. Likewise the span of edge e_{ij} is given by $(p_i \otimes p_j)^\perp$. Since c_{ij} belongs to both P_{ij} and e_{ij} , the fact that P_{ij} and e_{ij} are perpendicular is equivalent to the equation:

$$(c \otimes c_{ij}) * (p_i \otimes p_j) = 0.$$

Identities 3.1-3.4 imply this is equivalent to the equation:

$$c * ((c_{ij} * p_i)p_j - (c_{ij} * p_j)p_i) = 0.$$

Hence, Equation 4.1 holds for simplex $\{1, 2, 3\}$ if and only if there is a nontrivial solution c to the system:

$$\begin{aligned} c * ((c_{12} * p_1)p_2 - (c_{12} * p_2)p_1) &= 0 \\ c * ((c_{23} * p_2)p_3 - (c_{23} * p_3)p_2) &= 0 \\ c * ((c_{31} * p_3)p_1 - (c_{31} * p_1)p_3) &= 0 \end{aligned}$$

This system can be reformulated as a matrix equation

$$\begin{bmatrix} ((c_{12} * p_1)p_2 - (c_{12} * p_2)p_1)^T \\ ((c_{23} * p_2)p_3 - (c_{23} * p_3)p_2)^T \\ ((c_{31} * p_3)p_1 - (c_{31} * p_1)p_3)^T \end{bmatrix} \cdot J \cdot c = 0$$

that has a nontrivial solution if and only if the determinant of the first matrix is zero. Expanding that determinant and canceling the (nonzero) factors of $\det(p_1, p_2, p_3)$ that arise yields Equation 4.2. The last statement follows easily. \square

This proposition motivates the hyperbolic case of Definition 7.

Remark 1. *One can gain insight into how the Euclidean and hyperbolic compatibility conditions are related by comparing Equation 1.1 and Equation 1.2 for small d_{ij} in the same way one compares the Euclidean Pythagorean Theorem with the hyperbolic version, $\cosh(c) = \cosh(a)\cosh(b)$.*

5. CONFORMAL VARIATIONS OF HYPERBOLIC TRIANGLES

Various formulations of conformal variations of hyperbolic triangulations of surfaces have been studied in [44], [35], [12], [10], [41], [26], [1], [48]. We present a unified approach from the perspective of the metric triangulations as defined above.

5.1. Motivation and variation formula. Suppose we wanted to generate a metric from weights assigned to vertices, so that $d_{ij} = d_{ij}(f_i, f_j)$ for some function f on the vertices. If this our starting point for conformal structure, in order to compute conformal variations, we will consider what happens to the metric on a triangle $\{1, 2, 3\}$ when the conformal parameter f_3 changes but the other two do not, i.e., $\delta f_1 = \delta f_2 = 0$. We will call this a f_3 -conformal variation in this section.

The next two propositions analyze the configuration shown in Figure 2. We assume throughout that v_1, v_2, v_3 are linearly independent in \mathbb{R}^3 , with $v_i * v_i = -1$.

Proposition 7. *Under an f_3 -conformal variation:*

$$\begin{aligned} v_1 * \delta v_3 &= -\sinh \ell_{13} \frac{\partial \ell_{13}}{\partial f_3} \delta f_3 \\ v_2 * \delta v_3 &= -\sinh \ell_{23} \frac{\partial \ell_{23}}{\partial f_3} \delta f_3 \\ v_3 * \delta v_3 &= 0 \end{aligned}$$

Proof. Bilinearity of the Lorentzian inner product implies:

$$\delta(v_1 * v_3) = v_1 * \delta v_3.$$

However, since $v_1 * v_3 = -\cosh(\ell_{13})$, we can also write:

$$\delta(v_1 * v_3) = -\sinh \ell_{13} \frac{\partial \ell_{13}}{\partial f_3} \delta f_3.$$

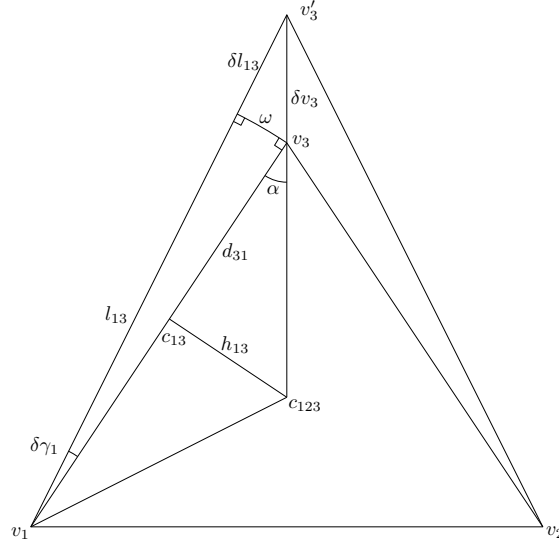


FIGURE 2. Variation of a Hyperbolic Triangle

Hence

$$v_1 * \delta v_3 = -\sinh \ell_{13} \frac{\partial \ell_{13}}{\partial f_3} \delta f_3.$$

We get the formula for $v_2 * \delta v_3$ similarly. Finally, since $v_3 * v_3 = -1$:

$$0 = \delta(v_3 * v_3) = 2v_3 * \delta v_3.$$

□

The following proposition makes precise what we mean by the colloquial statement that conformal variations give good angle variations.

Proposition 8. *Let c_{123} denote the center of the triangle specified by the vertices v_i and the (compatible) partial edge lengths d_{ij} . Suppose further that the edge centers on edges $\{1, 3\}$ and $\{2, 3\}$ are timelike. Then under a f_3 -conformal variation, the points v'_3, v_3 and c_{123} lie on a line in \mathbb{H} if and only if $\frac{\partial \ell_{ij}}{\partial f_i} = (\tanh d_{ij})F(f_i)$, for some function $F(f_i)$.*

Proof. Without loss of generality, assume $c_{123} * c_{123} = \pm 1$. Consider the geodesic through v_3 and c_{123} . As a set, this geodesic can be described by $\mathbb{H} \cap \text{Span}(v_3, c_{123})$, a characterization we will use to parametrize the geodesic by arclength as $\cosh(t)v_3 + \sinh(t)u$ for some $u \in v_3^\perp \cong T_{v_3}\mathbb{H}$. Specifically, Proposition 5 implies:

$$u = \frac{c_{123} + (v_3 * c_{123})v_3}{\sqrt{c_{123} * c_{123} + (v_3 * c_{123})^2}}$$

The points v'_3, v_3 , and c_{123} lie on a geodesic if and only if u and δv_3 are collinear. The three numbers $\{\delta v_3 * v_i\}_{i=1}^3$ completely characterize the vector $\delta v_3 \in v_3^\perp \cong T_{v_3}\mathbb{H}$. Hence, u and δv_3 are collinear if and only if there exists $\lambda \in \mathbb{R}$ such that $u * v_i = \lambda \delta v_3 * v_i$ for $i = 1, 2, 3$. We already know $v_3 * u = v_3 * \delta v_3 = 0$, so only $v_1 * u$ and $v_2 * u$ require consideration.

Consider our equation for u . The scalar in the denominator will appear in both $v_1 * u$ and $v_2 * u$. To simplify our notation, we will write $\lambda_3 := (c_{123} * c_{123} + (v_3 * c_{123})^2)^{-1/2}$. Now

$$v_1 * u = \lambda_3(c_{123} * v_1 + (v_3 * c_{123})(v_3 * v_1))$$

and we can apply the Generalized Law of Cosines (Proposition 3) to the triangle $\{v_1, v_3, c_{123}\}$ in order to rewrite this equation as

$$\begin{aligned} v_1 * u &= \lambda_3 \|v_1 \otimes v_3\| \|v_3 \otimes c_{123}\| \cos(\alpha) \\ &= \lambda_3 \sinh(\ell_{13}) \|v_3 \otimes c_{123}\| \cos(\alpha). \end{aligned}$$

Next consider the right triangle with vertices $\{c_{123}, c_{13}, v_3\}$. By Corollary 2, we have

$$\|v_3 \otimes c_{123}\| \cos(\alpha) = -(v_3 * c_{123}) \tanh(d_{31}).$$

A final substitution into our equation for $v_1 * u$ implies

$$v_1 * u = -(\lambda_3 \cdot v_3 * c_{123}) \sinh(\ell_{13}) \tanh(d_{31}).$$

A similar argument for v_2 yields

$$v_2 * u = -(\lambda_3 \cdot v_3 * c_{123}) \sinh(\ell_{23}) \tanh(d_{32}).$$

From Proposition 7, we know that for $k = 1, 2$

$$v_k * \delta v_3 = -\sinh \ell_{k3} \frac{\partial \ell_{k3}}{\partial f_3} \delta f_3.$$

Comparing these two equations, we see that there exists $\lambda \in \mathbb{R}$ so that $v_k * u = \lambda v_k * \delta v_3$ if and only if there exists a smooth function $F(f_3)$ for which $\frac{\partial \ell_{k3}}{\partial f_3} = \tanh(d_{3k}) F(f_3)$. \square

Proposition 8 motivates the hyperbolic case of Definition 8, where we have chosen to simplify to parameters that make F equal to the constant function 1.

We will now study how the angles change under a conformal variation. First we see the following.

Theorem 7. *Given a conformal structure, then for any simplex $\{i, j, k\}$*

$$(5.1) \quad \frac{\partial \gamma_i}{\partial f_j} = \frac{1}{\cosh d_{ji}} \frac{\tanh^\beta h_{ij}}{\sinh \ell_{ij}}$$

$$(5.2) \quad \frac{\partial \gamma_i}{\partial f_i} = -\frac{\partial A_{ijk}}{\partial f_i} - \frac{\partial \gamma_j}{\partial f_i} - \frac{\partial \gamma_k}{\partial f_i}$$

where β is 1 if c_{ijk} is timelike and -1 if c_{ijk} is spacelike.

Proof. For simplicity, we shall consider the problem for a single simplex $\{1, 2, 3\}$ labeled as in Figure 2, with $i = 1, j = 3$. We will address the case where c_{123} is timelike; the case where c_{123} is spacelike is similar. Once 5.1 is proven, 5.2 follows immediately because of the area formula for a hyperbolic triangle:

$$A_{123} = \pi - \gamma_1 - \gamma_2 - \gamma_3.$$

Because the variation is conformal, $\delta \ell_{13} = \tanh d_{31} \delta f_3$. Using the formula for a segment of a circle in the hyperbolic plane, we have $\omega = \delta \gamma_1 \sinh \ell_{13}$.

By Proposition 8, under a conformal variation v_3, v'_3 and c_{123} are collinear. Consequently, the angle adjacent to v_3 in the triangle with side lengths ω , δl_{13} and δv_3 is $\pi/2 - \alpha$. This, together with the formulas in Proposition 4, allows us to write:

$$\begin{aligned}\tan(\alpha) &= \frac{\tanh h_{13}}{\sinh d_{31}}, \\ \cot(\alpha) &= \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\tanh \delta l_{13}}{\sinh \omega}, \\ \frac{\tanh h_{13}}{\sinh d_{31}} &= \frac{\sinh \omega}{\tanh \delta l_{13}} = \frac{\sinh(\delta \gamma_1 \sinh \ell_{13})}{\tanh(\delta f_3 \tanh d_{31})}.\end{aligned}$$

Using the Taylor series for \sinh and \tanh , we have:

$$\frac{\tanh h_{13}}{\sinh d_{31}} = \frac{\delta \gamma_1 \sinh \ell_{13} + O(\delta \gamma_1^3)}{\delta f_3 \tanh d_{31} + O(\delta f_3^3)},$$

and hence,

$$\frac{\delta \gamma_1}{\delta f_3} = \frac{1}{\cosh d_{31}} \frac{\tanh h_{13}}{\sinh \ell_{13}} \left(\frac{1 + O(\delta f_3^2)}{1 + O(\delta \gamma_1^2)} \right).$$

□

We can also compute the variation of area explicitly.

Proposition 9. *Given a conformal structure, then for any simplex $\{i, j, k\}$ with area A_{ijk} .*

$$(5.3) \quad \frac{\partial A_{ijk}}{\partial f_k} = \frac{\partial \gamma_i}{\partial f_k} (\cosh \ell_{ik} - 1) + 2 \frac{\partial \gamma_j}{\partial f_k} (\cosh \ell_{jk} - 1).$$

In particular, if the derivatives $\partial \gamma_i / \partial f_k$ are positive whenever $k \neq i$, then the derivative of the area is positive.

Proof. This follows from the formula for the area of a sector of circle as a function of the radius for a hyperbolic surface, since in Figure 2 we find that the area of each of the small triangles is higher order, leaving only the areas of the skinny triangles in the picture. □

5.2. Characterization of discrete conformal structures. The proof of the hyperbolic case of Theorem 4 is similar to the proof of the Euclidean case, though the calculation is a bit harder in hyperbolic background.

Proof of the hyperbolic case of Theorem 4. We first note the following:

$$(5.4) \quad \frac{\partial}{\partial f_i} \cosh \ell_{ij} = \cosh \ell_{ij} - \frac{\cosh d_{ji}}{\cosh d_{ij}},$$

$$(5.5) \quad \frac{\partial}{\partial f_j} \cosh \ell_{ij} = \cosh \ell_{ij} - \frac{\cosh d_{ij}}{\cosh d_{ji}}.$$

A straightforward calculations gives that

$$\left(\frac{\cosh^2 d_{ij}}{\cosh^2 d_{ji}} \frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j} \right) \log \frac{\cosh^2 d_{ij}}{\cosh^2 d_{ji}} = 2 \left(\frac{\cosh^2 d_{ij}}{\cosh^2 d_{ji}} - 1 \right)$$

or if $H = \log \frac{\cosh^2 d_{ij}}{\cosh^2 d_{ji}}$ then

$$\left(e^H \frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j} \right) H = 2 (e^H - 1).$$

Since

$$\frac{\partial^2 H}{\partial f_i \partial f_j} = 0$$

it follows that

$$e^H \frac{\partial^2 H}{\partial f_i^2} + e^H \left(\frac{\partial H}{\partial f_i} \right)^2 = 2e^H \frac{\partial H}{\partial f_i}$$

and

$$e^{-H} \frac{\partial^2 H}{\partial f_j^2} - e^{-H} \left(\frac{\partial H}{\partial f_j} \right)^2 = 2e^{-H} \frac{\partial H}{\partial f_j}.$$

One can then easily solve this ODE to obtain that:

$$\frac{\partial H}{\partial f_i} = 2 \frac{a_{ij} e^{2f_i}}{1 + a_{ij} e^{2f_i}}$$

for some constant a_{ij} and

$$\frac{\partial H}{\partial f_j} = -2 \frac{a_{ji} e^{2f_j}}{1 + a_{ji} e^{2f_j}}$$

for some constant a_{ji} . It follows that

$$(5.6) \quad \frac{\cosh^2 d_{ij}}{\cosh^2 d_{ji}} = D \frac{1 + a_{ij} e^{2f_i}}{1 + a_{ji} e^{2f_j}}.$$

We can now use Equation 5.4 to see that

$$\begin{aligned} \cosh \ell_{ij} - \frac{\partial}{\partial f_i} \cosh \ell_{ij} &= \frac{1}{D} \left(\frac{1 + a_{ij} e^{2f_i}}{1 + a_{ji} e^{2f_j}} \right)^{-1/2} \\ \cosh \ell_{ij} - \frac{\partial}{\partial f_j} \cosh \ell_{ij} &= D \left(\frac{1 + a_{ij} e^{2f_i}}{1 + a_{ji} e^{2f_j}} \right)^{1/2} \end{aligned}$$

and so we find that $D = 1$ and

$$(5.7) \quad \cosh \ell_{ij} = \sqrt{(1 + a_{ji} e^{2f_j})(1 + a_{ij} e^{2f_i})} + \eta_{ij} e^{f_i + f_j}$$

for some constant η_{ij} .

The compatibility condition (1.2) implies that $\log \frac{\cosh d_{ij}}{\cosh d_{ji}} + \log \frac{\cosh d_{ki}}{\cosh d_{ik}}$ is independent of f_i and so we can use Equation 5.6 to see that $a_{ij} = a_{ik}$ and so we can define $\alpha_i = a_{ij} = a_{ik}$.

It follows that

$$\begin{aligned} \tanh d_{ij} &= \frac{1}{\sinh \ell_{ij}} \frac{\partial}{\partial f_i} \cosh \ell_{ij} \\ &= \frac{\alpha_i e^{2f_i}}{\sinh \ell_{ij}} \sqrt{\frac{1 + \alpha_j e^{2f_j}}{1 + \alpha_i e^{2f_i}}} + \frac{\eta_{ij} e^{f_i + f_j}}{\sinh \ell_{ij}}. \end{aligned}$$

Finally, we can use Equation 5.6 again to write $2 \log \frac{\cosh d_{ij}}{\cosh d_{ji}}$ in terms of the coefficients determined in the two triangles adjacent to edge $\{i, j\}$ and differentiate to see that the α_i derived in each triangle must be equal. It then follows from Equation 5.7 that the η_{ij} derived in each triangle must be equal as well. \square

5.3. Variational formulation for curvature. While the formula (1.9) is not symmetric in i and j , we can reparametrize to get a symmetric variation formula. Notice that Equation 5.6 (recall that we proved $D = 1$) implies that

$$\frac{\sqrt{1 + \alpha_i e^{2f_i}}}{\cosh d_{ij}} = \frac{\sqrt{1 + \alpha_j e^{2f_j}}}{\cosh d_{ji}}.$$

If we take new coordinates $u_i = u_i(f_i)$ such that

$$\frac{\partial f_i}{\partial u_i} = \sqrt{1 + \alpha_i e^{2f_i}}$$

then we have the symmetry

$$\frac{\partial \gamma_i}{\partial u_j} = \frac{\partial \gamma_j}{\partial u_i}.$$

Remark 2. The function $u_i(f_i)$ can be computed explicitly. It is not hard to see that if $\alpha_i = 0$ then $u_i = f_i$ and if not then

$$u_i = \frac{1}{2} \log \left| \frac{\sqrt{1 + \alpha_i e^{2f_i}} - 1}{\sqrt{1 + \alpha_i e^{2f_i}} + 1} \right|.$$

If $\alpha_i < 0$ then this is

$$-\tanh u_i = \sqrt{1 + \alpha_i e^{2f_i}}$$

and if $\alpha_i > 0$ then this is

$$-\coth u_i = \sqrt{1 + \alpha_i e^{2f_i}}.$$

Compare to the formulations in [26], [1], and [48].

It then follows that for a triangle $t = \{1, 2, 3\}$ the following form is closed:

$$(5.8) \quad \omega_t = \sum_{i=1}^3 \gamma_i du_i.$$

We can now integrate to get a function on the whole triangulation, where we fix some \bar{u} :

$$(5.9) \quad F(u) = 2\pi \sum_i u_i - \sum_t \int_{\bar{u}}^u \omega_t.$$

Theorem 8. The function F has the property that

$$\frac{\partial F}{\partial u_i} = K_i.$$

Furthermore, if all $d_{ij} > 0$ and $h_{ij} > 0$ then this function is strictly convex.

Proof. The first statement follows from the definition. The second follows from the facts that in a triangle $\{1, 2, 3\}$,

$$\begin{aligned} \frac{\partial \gamma_i}{\partial u_j} &\geq 0 \\ \left| \frac{\partial \gamma_i}{\partial u_i} \right| &> \frac{\partial \gamma_i}{\partial u_j} + \frac{\partial \gamma_i}{\partial u_k} \end{aligned}$$

for $\{i, j, k\} = \{1, 2, 3\}$ since

$$\frac{\partial A_{123}}{\partial f_i} > 0$$

by Proposition 9. It follows that the matrix of partial derivatives is diagonally dominant. \square

6. SPHERICAL GEOMETRY

The arguments presented in the case of hyperbolic background geometry can be adjusted for the case of spherical background geometry. Essentially, this occurs because in the hyperbolic case we are studying properties of the Lorentzian inner product $*$, while in spherical geometry we study analogous properties of the Euclidean inner product. Because the definitions and arguments in the spherical case are so similar to those of previous sections, we will only state the main results in the spherical case.

To work in the spherical case, we work with the usual dot product \cdot on \mathbb{R}^3 . Geodesics on the sphere correspond to planes in \mathbb{R}^3 and so given a triangle $\{i, j, k\}$ in the sphere and a pre-metric, a given embedding induces planes P_{ij} , etc. through edge centers and the condition for inducing a duality structure is

$$(6.1) \quad P_{ij} \cap P_{jk} = P_{jk} \cap P_{ki} = P_{ki} \cap P_{ij}$$

As in the hyperbolic case, this corresponds to a compatibility condition on the partial edge lengths.

Proposition 10. *Suppose d is a piecewise spherical pre-metric. Equation 6.1 holds if and only if the following compatibility equation*

$$(6.2) \quad (p_i \cdot c_{ij})(p_j \cdot c_{jk})(p_k \cdot c_{ki}) = (p_i \cdot c_{ki})(p_j \cdot c_{ij})(p_k \cdot c_{jk})$$

is satisfied for every simplex $\{i, j, k\}$. Equation 6.2 has the following equivalent formulation:

$$\cos(d_{ij}) \cos(d_{jk}) \cos(d_{ki}) = \cos(d_{ji}) \cos(d_{kj}) \cos(d_{ik}).$$

We can also look at discrete conformal structures. The angle variation theorem takes the following form.

Theorem 9. *Given a conformal structure, then for any simplex $\{i, j, k\}$:*

$$(6.3) \quad \frac{\partial \gamma_i}{\partial f_j} = \frac{1}{\cos d_{ji}} \frac{\tan h_{ij}}{\sin \ell_{ij}}$$

$$(6.4) \quad \frac{\partial \gamma_i}{\partial f_i} = \frac{\partial A_{ijk}}{\partial f_i} - \frac{\partial \gamma_j}{\partial f_i} - \frac{\partial \gamma_k}{\partial f_i}.$$

Note that although the heights h_{ij} require choosing one of the two possible centers, the term $\tan h_{ij}$ does not depend on this choice, since choosing the other center leads to heights $h'_{ij} = -(\pi - h_{ij})$ and so $\tan h'_{ij} = \tan h_{ij}$.

We can also compute the variation of area explicitly.

Proposition 11. *Given a spherical conformal structure, then for any simplex $\{i, j, k\}$ with area A_{ijk} , we have*

$$\frac{\partial A_{ijk}}{\partial f_k} = \frac{\partial \gamma_i}{\partial f_k} (1 - \cos \ell_{ik}) + \frac{\partial \gamma_j}{\partial f_k} (1 - \cos \ell_{jk}).$$

Using this theorem and the definition of a spherical conformal structure, one can derive the spherical case of Theorem 4. As in the hyperbolic case, it is desirable

to change from the variables f_i to variables $u_i = u_i(f_i)$, so that one can recognize that $\partial\gamma_i/\partial u_j = \partial\gamma_j/\partial u_i$. The variables u_i are given by

$$\frac{\partial f_i}{\partial u_i} = \sqrt{1 - \alpha_i e^{2f_i}}$$

Finally, we may define closed forms ω_t and a function F as in Equations 5.8 and 5.9. We have the following analog to Theorem 8.

Theorem 10. *The function F has the property that*

$$\frac{\partial F}{\partial u_i} = K_i.$$

Notice that we do not have a corresponding notion of convexity for this functional, as we do in the cases of Euclidean and hyperbolic backgrounds.

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